Neural Networks for Nonlinear Fractional Programming

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Abstract - This paper presents a neural network for solving non-linear minimax multiobjective fractional programming problem subject to nonlinear inequality constraints. Neural model is designed for optimization with constraints condition. Methodology is based on the lagrange multiplier with saddle point optimization.

Key words: Multiobjective, Fractional programming, saddle point, Lagrange multiplier, variational inequality, projection.

1. Introduction

Optimization problems arise in a wide variety of scientific and engineering applications including signal processing, system identification, filter design, function approximation, regression analysis and so on.

Here, we perform a rigorous analysis of a neural network for solving non-linear fractional programming problems and present a saddle point optimality theory for the minimax fractional programming problem. Various numerical procedures have been presented over decades for solving linear and nonlinear optimization problems. Tank and Hop field [5] in 1986 first proposed a neural network for linear programming.

Kennedy and Chua [20,21] extended and improved the tank and Hopfield network by using penalty method for solving nonlinear programming problem. Variety of attempts to avoid using penalty parameters have been made. Radriguez – Vazquez et al.[22] proposed a switched – capacitor neural network for solving a class of constrained non-linear convex optimization problems. Neural network models for optimization problem have been investigated intensively since the pioneer work of Hop field see [18,6,2,4] , for the eigen value problem, Xia [24] gave a promising neural network model which was proved to have global convergence with respect to the problems feasible set. Xia and Wang [8] gave a general neural network model designing methodology which put together way gradient based network models for solving the convex programming problems with globally convergent stability . Neural network for quadratic and nonlinear optimization with interval constraints were developed by Bouzerdorm, pattison [23] and Liang, Wang [3] and others [1],[10]-[17]. All there neural networks can be classified into the following three types:

(1) The gradient- based models
(2) The penalty function based models
(3) The projection based models

Nonlinear fractional programming does not belong to convex optimization problems and how to construct a good performance neural network model to solve this optimization problem becomes a challenge now since.

Fractional programming is a nonlinear programming method that has known increasing exposure recently and its importance in solving concrete problems is steadily increasing. Also non-linear optimization models describe practical problems much better than the linear optimization does. The fractional programming problems are particularly useful in the solution of economic problems in which various activities use certain resources in various proportions while the objective is to optimize a certain indicator, usually the most favorable return – on – allocation ratio subject to the constraint imposed on the availability of goods. Pal & Gupta [9] 2008 presented a Goal programming approach for solving interval valued multi objective fractional programming problems using Genetic Algorithm. Zhang & Feng [7] developed Neuro dynamic analysis for a class of nonlinear fractional programming. Wen and Wu [15] solved a continuous- time linear fractional programming problems by using the parametric method Neural circuit design techniques and related characteristics analysis is now becoming a typically challenging undertaking. Motivated by this idea, this paper is organized as follows.

In this section 2, we formulate the multi objecting non-linear fractional programming problem and its duality. In section 3, some illustrative examples are presented. In section 4, Neural model is primarily designed for optimization with constraints condition. The methodology is based on the lagrange multiplier with saddle point which satisfies the optimality and conclusion part in section 5 are cited.

Section -2 : We consider the following problems:

$$(P) \quad \min_{x \in X} \max_{1 \le i \le p} \left(f_i(x) \mid h_i(x) \right)$$

$$\text{Subject to } g(x) \le 0, x \in X \subseteq R^n$$

where f_i , h_i , i = 1,2,…p are real valued functions defined on X, each h_i is strictly positive and g = (g_1, g_2 ……g_m) , where each g_j is a real valued function defined on x.

To develop the optimality conditions, consider the following auxiliary problem:

(P_e) $\min_{x\in X} \max_{1\le i\le p} \{f_i(x) - e\, h_i(x)\}$

Subject to $g(x)\le 0$, $x \in X \subseteq R^n$

$Le\ (x,u) = \max_{1\le i\le p} (f_i(x) - e\ h_i(x)) + \sum_{j=1}^{m} u_j g_j(x)$ for fixed real number e and for all $x\in X$ and $u \in R^m$.

The lagrangian dual of (P) is defined as follows:

$$\max_{u\ge o}\left[\left\{\inf_{x\in X}\left(\max_{1\le i\le p} f_i(x) - e h_i(x)\right) + \sum_{j=1}^{m} u_j\, g_j(x)\right\}\right]$$

For a fixed e and for each $x\in X$ & $u \in R^m$

Definition 1: If there exists $x^* \in X$, $u^* \in R^m$ $u^* \ge 0$ such that

$$Le(x^*, u) \le Le\ (x^*, u^*) \le Le\ (x, u^*)$$

for all $x \in X$ and for all $u \in R^m$, $u \ge 0$, then (x^*, u^*) is called a saddle point of problem (P_e).

Definition 2: A pair (x^*, u^*) with $x^* \in X$, $u^* \in R^m$ is said to satisfy the optimality conditions for the problem (p_e) if and only if the following four conditions are satisfied

(a) x^* minimizes $Le\ (x, u^*)$
(b) $u^*\, g\ (x^*) = 0$
(c) $g\ (x^*) \le 0$
(d) $u^* \ge 0$

Definition 3: A point $u^* \in R^m$ is said to be an optimal multiplier of problem (P_e) if and only if there exists an $x^* \in X$ such that (x^*, u^*) satisfies optimality condition of definition 2.

Main Results:

If (x^*, u^*) satisfies the optimality conditions for (P_e) with $e^* = \max_{1\le i\le p} f_i(x_i)/h_i(x^*)$, then

(x^*, u^*) also satisfies the optimality conditions for (P)

Result 1:

For any

$$u\in U,\ x\in X,\ \inf_{x\in X} Le\ (x,u) = \inf_{x\in X}\left[\max_{1\le i\le p}\left(f_i(x) - e\ hi(x)\right) + \sum_{j=1}^{m} u_j\, g_j(x)\right]$$
$$\le \max_{1\le i\le p}\left(f_i(x)/h_i(x)\right)$$

Proof: For any $u \in U$, $x \in X$

$$\inf_{x\in X} \text{Le}\ (x,\ \text{u}) = \inf_{x\in X}\left\{\max_{1\le i\le p}\left(f_i(x) - e\, h_i(x)\right) + \sum_{j=1}^{m} u_j\, g_j(x)\right\}$$
$$\le \max_{1\le i\le p}\left(f_i(x) - e\, h_i(x)\right) + \sum_{j=1}^{m} u_j g_j(x)$$
$$\le \max_{1\le i\le p}\left(f_i(x)/e h_i(x)\right) \text{ since } g_j(x)\le 0 \text{ and } u_j \ge 0 \text{ for all j.}$$
$$\le \max_{1\le i\le p}\left(f_i(x)/h_i(x)\right),\ h_i(x) > 0\ \textit{for all } i$$

Hence the conclusion follows:

Result 2 : If u^* is an optimal multiplier for (P_e) then for every optimal solution x^* of (P_e) , (x^*, u^*) satisfies the optimality conditions for (P_e) .

Proof : Since u^* is an optimal multiplier for (P_e), there exists $x \in X$ such that it satisfies

(i) minimizes of Le (x^0 , u^*)
(ii) $u^* g(x^0) = 0$
(iii) $g(x^0) \le 0$
(iv) $u^* \ge 0$

From (ii) , (iii) and (iv) conditions it follows that $u^* g(x^0) = 0,\ \forall\ j$

Now for fixed e,

$$\max_{1\le i\le p}\left(f_i\ (x^*) \text{-} e h_i\ (x^*)\right)$$
$$= \inf_{x\in X}\max_{1\le i\le p}\left(f_i\ (x^*) \text{-} e h_i\ (x)\right)$$
$$\le \max_{1\le i\le p}\left(f_i(x^0) - e h_i(x^0)\right)$$
$$\le \max_{1\le i\le p}\left(f_i(x^0) - e h_i(x^0)\right) + \sum_{j=1}^{m} u_j^*\ g_j(x^0)$$
$$= Le(x^0, u^*) \le L(x^*, u^*)$$
$$= \max_{1\le i\le p}\left(f_i(x^*) - e\, h_i(x^*)\right) + \sum_{j=1}^{m} u_j^* . g_j(x^*)$$
$$\le \max_{1\le i\le p}\left(f_i\ (x^*) - e\ h_i\ (x^*)\right)$$

Since $g(x^*) \le 0$, $u^* \ge 0$, it follows that $Le(x^*, u^*) \le Le\ (x, u^*)\ \ \forall\ x \in X$

Hence (x^*, u^*) satisfies the optimality conditions for (P_e).

Result 3 : A pair (x^*, u^*) satisfies the optimality conditions (i) to (iv) of definition 2 if and only if it satisfies the following conditions

(i) x^* is an optimal solution (P_e)
(ii) u^* is an optimal solution of D
(iii) $\max_{1\le i\le p} (fi\ (x^*)\ \ e\ hi\ (x^*)) = \inf_{x\in X} Le\ (x, u^*)$

Proof: Now (x^*, u^*) satisfies the optimality conditions of (i) to (iv) of definition – 2. Then for any $u \in R^m$, $u \ge 0$, $u\ g(x^*) \le 0$

$$\text{Le}\ \ (x^*, u) = \max_{1\le i\le p}\left(f_i(x^*) - e\, h_i\ (x^*)\right) + \sum_{j=1}^{m} u_j g_j(x^*) \quad \text{for all fixed } e$$
$$\le \max_{1\le i\le p}\left(f_i(x^*) - e\ h_i\ (x^*)\right), \text{ since } u\ g(x^*) \le 0$$
$$= \max_{1\le i\le p}\left(f_i(x^*) - e\ h_i(x^*)\right) + \sum_{j=1}^{m} u_j^* g_j(x^*)$$
$$= Le(x^*, u^*)$$

From optimality conditions,

$$Le(x^*, u^*) \le Le\,(x, u^*)\ \forall\, x \in X \qquad (2)$$

Hence from equation (1) and (2) it follows that $Le\,(x^*, u) \le Le\,(x^*, u^*) \le Le\,(x, u^*)$

This implies that (x^*, u^*) is a saddle point.

Since (x^*, u^*) is a saddle point of Le (x, u^*),

then (x^*, u^*) is an optimal solution of (P_e). This proves the result-3 conditions (i). To establish conditions (ii),

$$\inf_{x\in X} Le(x,u) = \inf_{x\in X}\left\{\max_{1\le i\le p}\left(f_i(x) - eh_i(x)\right) + \sum_{j=1}^{m} u_j g_j(x)\right\}$$
$$\le \max_{1\le i\le p}\left(f_i(x^*) - eh_i(x^*)\right) + \sum_{j=1}^{m} u_j g_j(x^*)$$
$$\le \max_{1\le i\le p}\left(f_i(x^*) - eh_i(x^*)\right)$$
$$= \max_{1\le i\le p}\left(f_i(x^*) - eh_i(x^*)\right) + \sum_{j=1}^{m} u_j^* g_j(x^*)$$
$$= Le\,(x^*, u^*)$$
$$\le Le\,(x, u^*)$$
$$= \max_{1\le i\le p}\left(f_i(x) - eh_i(x)\right) + \sum_{j=1}^{m} u_j^* g_j(x)\ \forall\ x \in X$$

$$\Rightarrow \inf_{x\in X} Le\,(x, u) \le \inf_{x\in X}\left\{\max_{1\le i\le p}\left(f_i(x) - eh_i(x)\right) + \sum_{j=1}^{m} u_j^* g_j(x)\right\}$$
$$= \inf_{x\in X} Le\,(x, u^*)$$

$\Rightarrow u^*$ is an optimal solution of *(D)*.

This establishes the proof for (ii).

To prove (iii), we consider from equation (2) that

$$Le\,(x^*, u^*) = \inf_{x\in X} Le\,(x, u^*)$$

or,
$$\max_{1\le i\le p}\left(f_i(x^*) - eh_i(x^*)\right) + \sum_{j=1}^{m} u_j^* g_j(x^*)$$
$$= \inf_{x\in X} Le\,(x, u^*)$$

Since, $u_j^* g_j(x^*) = 0\ \forall\, j = 1,2,\ldots m,$

$$\max_{1\le i\le p}\left(f_i(x^*) - eh_i(x^*)\right) = \inf_{x\in X} Le\,(x, u^*)$$

To prove for converse part, we consider that it satisfies the conditions from (i) to (iii).

Then,

$$\inf_{x\in X}\left\{\max_{1\le i\le p}\left(f_i(x) - eh_i(x)\right) + \sum_{j=1}^{m} u_j^* g_j(x)\right\}$$
$$\le \max_{1\le i\le p}\left(f_i(x^*) - eh_i(x^*)\right) + \sum_{j=1}^{m} u_j^* g_j(x)$$
$$\le \max_{1\le i\le p}\left(f_i(x^*) - eh_i(x^*)\right) = \inf_{x\in X} Le\,(x, u^*)$$

Hence all the relations are equal.

$\Rightarrow x^*$ minimizes $Le\,(x, u^*)$ *and* $u^* g(x^*) = 0$

The optimamality conditions (iii) and (iv) follow form the feasibility of x^* *and* u^*

Result 4 : The following statements are equivalent

(a) (x^*, u^*) is a saddle point of Le (x, u)

(b) Conditions of Result – 2 hold

(c) Condition (i) to (iv) of definition -2 hold.

We observe that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a)

Result 5: Suppose that (P) has an optimal solution x^* *and that* (P_e^*) is stable where $e^* = \max_{1\le i\le p}\left(f_i(x^*)/h_i(x^*)\right)$

Thus (*D*) has an optimal solution and the optimal values of (*P*) and (*D*) are equal.

Section – 3:

Definition 4 : A feasible solution x^* *of* (P_e) is said to be an efficient solution of (P_e) if there does not exist any feasible solution x of (P_e) such that

$\left(f_i(x) - eh_i(x)\right) \le \left(f_i(x^*) - eh_i(x^*)\right)$ *for* $i = 1,2\ldots p$ & for fixed *e*.

and $\left(f_i(x) - eh_i(x)\right) < f_j\left((x^*) - eh_j(x^*)\right)$

for some *j* and for fixed *e*

If x^* is an efficient solution of (P_e) then x^* is an efficient solution of (*P*)

Definition 5: A feasible solution x^* of (P_e) is said to be properly efficient solution of (P_e) if it is an efficient solution of (P_e) and there exists a scalar $M > 0$ such that for some i and for some feasible *x*,

$$\left(f_i(x) - e\,h_i(x)\right) < f_i(x^*) - eh_i(x^*),$$
$$\Rightarrow \left(f_i(x^*) - eh_i(x^*)\right) - \left(f_i(x) - e\,h_i(x)\right)$$
$$\le M\left(f_j(x) - e\ h_j(x)\right) - \left(f_j(x^*) - eh_j(x^*)\right) \text{ for some j}$$

such that

$$\left(f_j(x) - e\,h_i(x)\right) > \left(f_i(x^*) - eh_i(x^*)\right)$$

If (x^*) is an properly efficient solution of (P_e) then (x^*) is an properly efficient solution (P).

Example :

(F_p) $\min_x\left(f_1(x) - eh_1(x), f_2(x) - eh_2(x)\right)$

subject to $g_j(x) \le 0$, $j = 1,2$

and $x \in X =]-2,\ 2[$

$f_1(x) = x^2 + 1$	$f_2(x) = 3 - x$
$h_1(x) = x^2 + 2$	$h_2(x) = x + 2$
$g_1(x) = x^2 - 1$	$g_2(x) = -x$

The feasible region is $[0,1]$. Let us take $x^* = 1 \in [0,1]$ and *e*=1 for fixed real number.

$$\left(f_1(x) - e\,h_1(x)\right) - \left(f_1(x^*) - e\,h_1(x^*)\right)$$
$$= (x^2 + 1 - x^2 - 2) - (2 - 3) = -1 + 1 = 0$$

Now we can prove that $\bar{x} = 1$ is a properly efficient solution of (F_p).

Section 4: Neural Model

The neurons in the network can be classified into two classes: variable neurons x and Lagrangian neurons λ, with regard to their role in searching for the solution.

In the dynamic process of the neural network, Lagrangian neurons lead the trajectory intot the feasible region while variable neurons decrease the Lagrangian function *L(x,λ)*. The decrease of the Lagrangian function x can be verified from the fact that along the trajectory of the network

$$\left.\frac{dL(x,\lambda)}{dt}\right|_{\lambda=cons\tan t} = \sum_{i=1}^{n}\frac{\partial L(x,\lambda)}{\partial x_i}\frac{dx_i}{dt} = -\sum_{i=1}^{n}\left(\frac{dx_i}{dt}\right)^2 \le 0 .$$

Let f,h,g be pseudoconvex functions. Hence $\frac{f}{h}$ is pseudoconvex i.e. $f_i(x)-eh_i(x)$ is pseudoconvex.

Since, g is pseudoconvex and $y\ge 0$ that implies $\left(f_i(x)-e\ h_i(x)\right)+y^t g(x)$ is pseudoconvex.

Also $(x-x^*)^t\left[\nabla\left(f_i(x^*)-e\ h_i(x^*)\right)+y^t\nabla g(x^*)\right]\ge 0$, this satisfies variational Inequality problem over Ω_0.

Hence the new projection neural network model is

$$P_{\Omega_0}\left(x^*-\alpha\left[\nabla\left(f_i(x^*)-e\ h_i(x^*)\right)+y^t\nabla g(x^*)\right]\right)=x^*$$

where, $X=\left\{x\in\Re^n \mid d\le x\le h\right\}$,

$P_x:\Re^n\to x$ is a projection operator and $\Omega_0=\left\{x\in\Re^n \mid g(x)\le 0\right\}$ is closed convex.

Section 5:

Conclusion: The paper proposes a new projection neural network model and theoretically guaranteed to solve variational inequality problems. The multiobjective minimax nonlinear fractional programming is defined and its optimality is derived by using its Lagrangian duality. The equilibrium points of the proposed neural network model are found to correspond to the Karush Kuhn Trcker point associated with the nonlinear fractional programming problem.

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